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Modular invariants detecting the cohomology of BF_4 at the prime 3

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Attributed to J F Adams is the conjecture that, at odd primes, the mod–p cohomology ring of the classifying space of a connected compact Lie group is detected by its elementary abelian p–subgroups. In this note we rely on Toda's calculation of $H^*(BF_4;\mathbb{F}_3)$ in order to show that the conjecture holds in case of the exceptional Lie group F_4. To this aim we use invariant theory in order to identify parts of $H^*(BF_4;\mathbb{F}_3)$ with invariant subrings in the cohomology of elementary abelian 3–subgroups of F_4. These subgroups themselves are identified via the Steenrod algebra action on $H^*(BF_4;\mathbb{F}_3)$.

55R40; 55S10, 13A50

1 Introduction

It has been known since the work of Borel [3, 2] that the rational cohomology of the classifying space of a compact and connected Lie group G is detected on its maximal torus T_G, and can actually be identified with the subalgebra of elements in the cohomology of the classifying space of T_G that are fixed by the natural action of the Weyl group W_G; that is, we can identify $H^*(BG;\mathbb{Q}) \cong H^*(BT_G;\mathbb{Q})^{W_G}$. Similar identifications hold for cohomology with coefficients in fields of prime charateristic as soon as this charateristic does not divide the order of the Weyl group.

A quick look at the mod p cohomology of classifying spaces of compact connected Lie groups at torsion primes (cf Mimura and Toda [12]) shows that restrictions to maximal tori usually have big kernels. In particular all odd degree elements can only be mapped trivially by the restriction to the maximal torus. We are then led to consider the restriction to elementary abelian subgroups. At odd primes, there is always a maximal one that consists of all elements of p–power order in the maximal torus, but in presence of torsion there are also elementary abelian subgroups which are non-toral; that is, not

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conjugate to a subgroup of the maximal torus. If $\mathcal{E}_p(G)$ is a set of representatives of all conjugacy classes of maximal elementary abelian p–subgroups, then the kernel of the restriction map

$$q_G\colon H^*(BG;\mathbb{F}_p) \longrightarrow \prod_{E\in\mathcal{E}_p(G)} H^*(BE;\mathbb{F}_p)$$

is nilpotent, according to Quillen [14], for any compact Lie group G and any prime p. Adams conjectured that q_G is actually a monomorphism if G is compact and connected and p is an odd prime.

In this note we rely on Toda's calculation [20] of $H^*(BF_4;\mathbb{F}_3)$ to show that q_G is a monomorphism in this case. Kono and Yagita [10] proved that q_G is a monomorphism for $G = PU(3)$ at the prime 3. This has been recently generalized by Vavpetič and Viruel [21] to $G = PU(p)$ at the prime p, for p odd. Mimura, Sambe, Tezuka, and Toda [11] have also obtained that the conjecture is true for $G = E_6$ at $p = 3$.

If $W_G(E)$ denotes the group of automorphisms of the elementary abelian subgroup E of G which are induced by conjugation in G, the restriction map has image in the invariant subring $H^*(BE;\mathbb{F}_p)^{W_G(E)}$. In section two we present the relevant invariant theory in order to have a description as algebras over the Steenrod algebra of these invariant rings for the elementary abelian 3–subgroups of F_4 that are involved in our calculations. These subgroups were identified by Rector [15, Section 7] by arguments based on work of Toda, and confirmed by Adams using geometric arguments. Taking Rector's calculations as starting point and comparing the Steenrod algebra action on $H^*(BF_4;\mathbb{F}_3)$ and on the invariant subrings $H^*(BT_{F_4};\mathbb{F}_3)^{W_{F_4}}$ and $H^*(BE;\mathbb{F}_3)^{W_{F_4}(E)}$, we obtain a precise description of q_{F_4} at the prime 3 in section three, and, in particular, that it is injective.

Part of the results presented here were announced in [6] but details remained unpublished at that time. Now, during the Conference in Algebraic Topology held in Hanoi in honor of Huỳnh Mui's 60th birthday, in August, 2004, I saw a renewed interest in the subject by M Kameko, M Mimura, and A Viruel, among others, which prompted me to submit this note to the conference proceedings.

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2 Modular invariants in $P(V) \otimes E(V)$

Let $\mathbb{F}_p$ be the field of p elements, for an odd prime p, and V a $\mathbb{F}_p$–vector space of dimension n. We denote by $P(V^*)$ the symmetric algebra on the dual vector space V^*. If $d\colon V^* \to dV^*$ is an isomorphism of $\mathbb{F}_p$–vector spaces and $E(dV^*)$ is the exterior algebra on dV^*, d extends uniquely to a derivation of the algebra

$$K(V^*) = P(V^*) \otimes E(dV^*) = P[x_1, \dots, x_n] \otimes E[dx_1, \dots, dx_n].$$

Let G be a finite subgroup of $GL(V)$, so that G acts on $P(V^*)$ via the transpose representation. This action can be extended to a differential algebra action on $K(V^*)$ in a unique way. We are interested in cases in which the fixed subalgebra is again a polynomial algebra on n generators

$$P(V^*)^G = P[\rho_1, \dots, \rho_n]\,.$$

In this case, according to a theorem of Serre, G is a pseudoreflection group, i.e., G is generated by elements that fix a codimension one subspace of V (see Bourbaki [4], Benson [1] and Smith [17]). If the order of G is not divisible by p, then $P(V^*)^G$ is a polynomial algebra if and only if G is a pseudoreflection group and in this case the ring of invariants of $K(V^*)$ is

$$K(V^*)^G = P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n],$$

(see Solomon [18], Benson [1] and Smith [17]).

In this section we will discuss the case in which $P(V^*)^G$ is a polynomial algebra but p divides the order of G. First, we will show a necessary and sufficient condition under which the above formula holds. In the case that this condition is not satisfied we show how to construct new invariants in $K(V^*)$. In the cases that we have checked, these new invariants provide a complete system of generators for $K(V^*)^G$. For $G = GL(V)$ or $G = SL(V)$ they can be compared with the system obtained by Mui [13].

The modules of relative invariants will play a fundamental role in our discussion. Recall that for a linear character of G, $\chi\colon G \to \mathbb{F}_p^*$, with $\chi(gh) = \chi(g)\chi(h)$, the $P(V^*)^G$–module of χ–relative invariants is defined

$$P(V^*)^G_\chi = \{\, q \in P(V^*) \mid g \cdot q = \chi(g)q,\ \forall g \in G \,\}\,.$$

If G is a pseudoreflection group, it turns out that this is a free $P(V^*)^G$–module of rank one, $P(V^*)^G_\chi = f_\chi P(V^*)^G$, for an element $f_\chi \in P(V^*)$ that can be written in a unique

way, up to scalar multiplication, as a product of forms in V^* (see Stanley [19] and Broto–Smith–Stong [7]).

If we write $d\rho_i = \sum_{i=1}^n a_{ij} dx_j$, the jacobian, $J = \det(a_{ij})_{i,j}$ is a non-trivial element of $P(V^*)$ (see Wilkerson [23]). Moreover, it is a $\det^{-1}$–relative invariant:

$$J \in P(V^*)^G_{\det^{-1}} = f_{\det^{-1}} \cdot P(V^*)^G.$$

It follows that $f_{\det^{-1}}$ always divides J.

Theorem 2.1 (Broto [5]) *Let $\mathbb{F}_p$ be the field of p elements, where p is an odd prime, and V a $\mathbb{F}_p$–vector space of dimension n. Assume that G is a finite subgroup of $GL(V)$ such that $P(V^*)^G = P[\rho_1, \dots, \rho_n]$, then*

$$K(V^*)^G = P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n]$$

if and only if $J = f_{\det^{-1}}$ (up to an invertible of $\mathbb{F}_p$).

Proof As in the characteristic zero case, the fact $J \neq 0$, implies that the morphism

$$P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n] \to K(V^*)$$

is always injective. We need to determine the cases in which all elements of $K(V^*)^G$ belong to the image; in other words, are expressible as algebraic combination of $\rho_1, \dots, \rho_n, d\rho_1, \dots, d\rho_n$.

If $I = (i_1, \dots, i_k)$ is an ordered sequence of integers: $0 < i_1 < \cdots < i_k \leq n$, we denote $d\rho_I = d\rho_{i_1} d\rho_{i_2} \cdots d\rho_{i_k}$ or $dx_I = dx_{i_1} dx_{i_2} \cdots dx_{i_k}$. With this notation, $K(V^*)$ is a free $P(V^*)$–module generated by $\{dx_I\}_I$, and so, if $FP(V^*)$ is the field of fractions of $P(V^*)$ and

$$FK(V^*) = FP(V^*) \otimes_{P(V^*)} K(V^*),$$

then $FK(V^*)$ is a $FP(V^*)$–vector space with base $\{dx_I\}_I$, and $\{d\rho_I\}_I$ form a base, too.

Assume first that $J = f_{\det^{-1}}$ up to an scalar. Choose an arbitrary element $w \in K(V^*)^G$. It may be written as a linear combination $w = \sum_I w_I d\rho_I$, with $w_I \in FP(V^*)^G$. We will show that each w_I lies in $P(V)^G$ so that $w \in P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n]$.

Choose a sequence I_0 of minimal length such that $w_{I_0} \neq 0$ and let I_0' be the complementary sequence, so that the expression

$$w d\rho_{I_0'} = w_{I_0} d\rho_{I_0} d\rho_{I_0'} = \pm w_{I_0} d\rho_1 \cdots d\rho_n = \pm w_{I_0} J dx_1 \cdots dx_n$$

is still an element of $K(V^*)^G$. But $dx_1 \cdots dx_n$ is invariant relative to det, hence $w_{I_0}J \in P(V^*)^G_{\det^{-1}} = f_{\det^{-1}}P(V^*)^G$, and so therefore $w_{I_0} \in P(V^*)^G$. We repeat the process with $w - w_{I_0}d\rho_{I_0}$ and we obtain inductively that all coefficients $w_I \in P(V^*)^G$.

Assume now that $J = \iota f_{\det^{-1}}$ for a positive degree polynomial $\iota \in P(V^*)^G$, then

$$w = \frac{d\rho_1 \cdots d\rho_n}{\iota} = f_{\det^{-1}}\, dx_1 \cdots dx_n$$

is an element in $K(V)^G$ but it does not belong to the subalgebra $P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n]$. $\square$

We have seen that whenever the jacobian J is different from $f_{\det^{-1}}$ in an essential way, we obtain a new invariant that does not belong to $P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n]$, by dividing $d\rho_1 \cdots d\rho_n$ by an invariant factor of $P(V^*)$. A similar argument applies to any $d\rho_I$; that is, we can divide each $d\rho_I$ by its *maximal invariant factor*, $B_I \in P(V^*)^G$, in order to obtain a new invariant: $M_I = \frac{1}{B_I}d\rho_I \in K(V)^G$.

More precisely, fix a sequence I, and write

$$d\rho_I = \sum_J a_J(I)dx_J, \quad a_J \in P(V^*)$$

where all sequences J have the same length as I. We define $A_I = \gcd(a_J(I))$ so that $d\rho_I = A_I \sum_J b_J(I)dx_J$ and the coefficients $b_J(I) \in P(V^*)$ have no common factor.

It turns out that A_I is relative invariant to a certain linear character χ_I of G. In fact, for any $g \in G$

$$\sum_J a_J(I)dx_J = d\rho_I = g(d\rho_I) = g\Big(A_I \sum_J b_J(I)dx_J\Big) = g(A_I)\sum_J b'_J(I)dx_J$$

hence $g(A_I)$ divides each $a_J(I)$ and so therefore, $g(A_I) = \chi_I(g)A_I$ for some element $\chi_I(g) \in \mathbb{F}_p^*$. This defines the character χ_I, and shows that $A_I \in P(V^*)^G_{\chi_I}$.

Since $P(V^*)^G_{\chi_I} = f_{\chi_I}P(V^*)^G$ for certain class $f_{\chi_I} \in P(V^*)^G_{\chi_I}$, we can define elements $B_I \in P(V^*)^G$ by the equation $A_I = B_I \cdot f_{\chi_I}$, and then $M_I = f_{\chi_I}\sum_J b_J(I)dx_J \in K(V^*)^G$ gives the factorization

$$d\rho_I = B_I \cdot M_I$$

with $M_I \in K(V^*)^G$ and $B_I \in P(V^*)^G$. Notice also that by construction we obtain relations

$$M_IM_J = \begin{cases} q_{I,J}M_{I\cup J} & \text{for some } q_{I,J} \in P(V^*)^G, \text{ if } J \cap I = \emptyset, \\ 0 & \text{if } J \cap I \neq \emptyset. \end{cases}$$

It seems reasonable to ask whether or not we have obtained a complete system of generators and relations for $K(V^*)^G$.

Question 2.2 *Is $K(V^*)^G$ a free $P(V^*)^G$–module generated by $\{M_I\}_I$, for every group $G \leq GL(V)$ for which $P(V^*)^G$ is a polynomial algebra?*

We have a positive answer in the cases which are involved in the mod 3 cohomology of BF_4.

Example 2.3 Let $\mathbb{F}_p$ be the field of p elements, p an odd prime, and assume $K(V^*) = P[x_1, x_2] \otimes E[dx_1, dx_2]$, and $G = GL_2(\mathbb{F}_p)$. The Dickson invariants are described, in terms of determinants of two by two matrices as

$$L_2 = \begin{vmatrix} x_1 & x_2 \\ x_1^p & x_2^p \end{vmatrix}, \qquad Q_{2,1} = \begin{vmatrix} x_1 & x_2 \\ x_1^p & x_2^p \end{vmatrix}^{-1} \cdot \begin{vmatrix} x_1 & x_2 \\ x_1^{p^2} & x_2^{p^2} \end{vmatrix}$$

and we have $P[x_1, x_2]^{GL_2(\mathbb{F}_p)} = P[L_2^{p-1}, Q_{2,1}]$ (see Dickson [9]). One then obtains

$$dL_2^{p-1} = -L_2^{p-2}(x_2^p dx_1 - x_1^p dx_2) \quad \text{and} \quad dQ_{2,1} = -L_2^{p-2}(x_2 dx_1 - x_1 dx_2).$$

Since $L_2^{p-2} = f_{\det^{-1}}$, we have

$$M_1 = dL_2^{p-1} \quad \text{and} \quad M_2 = dQ_{2,1}$$

with $B_1 = B_2 = 1$. On the other hand, $dL_1^{p-1} dQ_{2,1} = -L_2^{2p-3} dx_1 dx_2$, hence

$$M_{1,2} = -L_2^{p-2} dx_1 dx_2$$

with $M_1 M_2 = L_2^{p-1} M_{1,2}$. According to Mui [13], we know that

$$\{L_2^{p-1}, Q_{2,1}, M_1, M_2, M_{1,2}\}$$

is a full system of generators for $(P[x_1, x_2] \otimes E[dx_1, dx_2])^{GL_2(\mathbb{F}_p)}$.

Example 2.4 We describe the invariants of

$$H^*((\mathbb{Z}/p)^3; \mathbb{F}_p) = P[u_1, u_2, u_3] \otimes E[v_1, v_2, v_3],$$

$\deg v_i = 1$, $u_i = \beta v_i$ by the action of $SL_3(\mathbb{F}_p)$, for an odd prime p.

The Dickson invariants are the determinants

$$L_3 = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1^p & u_2^p & u_3^p \\ u_1^{p^2} & u_2^{p^2} & u_3^{p^2} \end{vmatrix}, \qquad \deg L_3 = 2\frac{p^3-1}{p-1}$$

$$Q_{3,2} = \frac{1}{L_3}\begin{vmatrix} u_1 & u_2 & u_3 \\ u_1^p & u_2^p & u_3^p \\ u_1^{p^3} & u_2^{p^3} & u_3^{p^3} \end{vmatrix}, \qquad \deg Q_{3,2} = 2(p^3-p^2)$$

$$Q_{3,1} = \frac{1}{L_3}\begin{vmatrix} u_1 & u_2 & u_3 \\ u_1^{p^2} & u_2^{p^2} & u_3^{p^2} \\ u_1^{p^3} & u_2^{p^3} & u_3^{p^3} \end{vmatrix}, \qquad \deg Q_{3,1} = 2(p^3-p)$$

The action of the Steenrod algebra on these elements is determined by

$$\begin{array}{lll} \mathcal{P}^1 L_3 = 0 & \mathcal{P}^1 Q_{3,2} = 0 & \mathcal{P} Q_{3,1} = L_3^2 \\ \mathcal{P}^p L_3 = 0 & \mathcal{P}^p Q_{3,2} = Q_{3,1} & \mathcal{P}^p Q_{3,1} = 0 \\ \mathcal{P}^{p^2} L_3 = Q_{3,2}L_3 & \mathcal{P}^{p^2} Q_{3,2} = -Q_{3,2}^2 & \mathcal{P}^{p^2} = -Q_{3,2}Q_{3,1} \end{array}$$

(see Dickson [9] and Wilkerson [22]). Since det is trivial on $SL_3(\mathbb{F}_p)$, $M_{1,2,3} = v_1v_2v_3$ is invariant. Steenrod operations can be used to find new invariants:

$$\begin{aligned} M_{2,3} &= \beta M_{1,2,3} = u_1v_2v_3 - u_2v_1v_3 + u_3v_1v_2, \\ M_{1,3} &= -\mathcal{P}^1 M_{2,3} = -u_1^p v_2v_3 - u_2^p v_1v_3 + u_3^p v_1v_2, \\ M_{1,2} &= \mathcal{P}^p M_{1,3} = -u_1^{p^2} v_2v_3 + u_2^{p^2} v_1v_3 - u_3^{p^2} v_1v_2, \\ M_3 &= \beta M_{1,3} = \begin{vmatrix} u_2 & u_3 \\ u_2^p & u_3^p \end{vmatrix} v_1 - \begin{vmatrix} u_1 & u_3 \\ u_1^p & u_3^p \end{vmatrix} v_2 + \begin{vmatrix} u_1 & u_2 \\ u_1^p & u_2^p \end{vmatrix} v_3, \\ M_2 &= \mathcal{P}^p M_3 = \begin{vmatrix} u_2 & u_3 \\ u_2^{p^2} & u_3^{p^2} \end{vmatrix} v_1 - \begin{vmatrix} u_1 & u_3 \\ u_1^{p^2} & u_3^{p^2} \end{vmatrix} v_2 + \begin{vmatrix} u_1 & u_2 \\ u_1^{p^2} & u_2^{p^2} \end{vmatrix} v_3, \\ M_1 &= \mathcal{P}^1 M_2 = \begin{vmatrix} u_2^p & u_3^p \\ u_2^{p^2} & u_3^{p^2} \end{vmatrix} v_1 - \begin{vmatrix} u_1^p & u_3^p \\ u_1^{p^2} & u_3^{p^2} \end{vmatrix} v_2 + \begin{vmatrix} u_1^p & u_2^p \\ u_1^{p^2} & u_2^{p^2} \end{vmatrix} v_3, \end{aligned}$$

and finally $\beta M_1 = L_3$. One can check that these are precisely the set of invariants

described above:

$$M_1 = dL_3, \qquad M_2 = \frac{1}{L_3}\, dQ_{3,2}, \qquad M_3 = \frac{1}{L_3}\, dQ_{3,1},$$

$$M_{1,2} = \frac{1}{{L_3}^2}\, dL_3\, dQ_{3,2}, \quad M_{1,3} = \frac{1}{{L_3}^2}\, dL_3\, dQ_{3,1}, \quad M_{2,3} = \frac{1}{{L_3}^2}\, dQ_{3,2}\, dQ_{3,1},$$

$$\text{and} \quad M_{1,2,3} = -\frac{1}{{L_3}^4}\, dL_3\, dQ_{3,2}\, dQ_{3,1}.$$

Again in this case, according to Mui [13],

$$\{L_3, Q_{3,1}, Q_{3,2}, M_1, M_2, M_3, M_{1,2}, M_{1,3}, M_{2,3}, M_{1,2,3}\}$$

forms a full system of generators for $H^*((\mathbb{Z}/p)^3;\mathbb{F}_p)^{SL_3(\mathbb{F}_p)}$.

3 On the cohomology of BF_4 at prime 3

In this section we will show how starting with the computation of $H^*(BF_4;\mathbb{F}_3)$ by Toda [20], one can obtain that this cohomology ring is detected on elementary abelian 3–subgroups. The argument goes through the description of $H^*(BF_4;\mathbb{F}_3)/\sqrt{0}$ by Rector [15].

For the convenience of the reader we present here Toda's description of the cohomology of the classifying space of the exceptional Lie group F_4 at $p = 3$. The Weyl group of F_4 contains the Weyl group of Spin_9, so that we have:

$$H^*(BT;\mathbb{F}_3)^{W_{F_4}} \subset H^*(BT;\mathbb{F}_3)^{W_{\mathrm{Spin}_9}} = P[p_1, p_2, p_3, p_4]\,,$$

where p_i are Pontrjagin classes. Toda first computed the invariant ring

$$H^*(BT;\mathbb{F}_3)^{W_{F_4}} = P[p_1, \bar{p_2}, \bar{p_5}, \bar{p_9}, \bar{p_{12}}]/(r_{15})$$

where

$$\begin{aligned}
\bar{p}_2 &= p_2 - p_1^2,\\
\bar{p}_5 &= p_4p_1 + p_3\bar{p}_2,\\
\bar{p}_9 &= p_3^3 - p_4p_3p_1^2 + p_3^2\bar{p}_2p_1 - p_4\bar{p}_2p_1^3,\\
\bar{p}_{12} &= p_4^3 + p_4^2\bar{p}_2^2 + p_4\bar{p}_2^4,\\
r_{15} &= \bar{p}_5^3 + \bar{p}_5^2\bar{p}_2^2p_1 - \bar{p}_{12}p_1^3 - \bar{p}_9\bar{p}_2^3,
\end{aligned}$$

and obtained elements x_4, x_8, x_{20}, x_{36}, x_{48} in $H^*(BF_4;\mathbb{F}_3)$ that restrict to p_1, $\bar{p}_2$, $\bar{p}_5$, $\bar{p}_9$, and $\bar{p}_{12}$, respectively in $H^*(BT;\mathbb{F}_3)^{W(F_4)}$.

Theorem 3.1 (Toda [20]) *$H^*(BF_4;\mathbb{F}_3)$ is an algebra generated by*

$$x_4,\quad x_8,\quad x_{20},\quad x_{36},\quad x_{48},$$
$$x_9=\beta x_8,\quad x_{21}=\beta x_{20},\quad x_{25}=\mathcal{P}^1 x_{21},\quad x_{26}=\beta x_{25},$$

with the relations

$$\begin{aligned}
x_9x_4 &= x_9x_8 = x_9^2 = x_{21}x_4 = x_{25}x_8\\
&= x_{21}x_{20} = x_{21}^2 = x_{25}x_{20} = x_{25}^2 = 0,\\
x_{21}x_8 &= x_{25}x_4 = -x_{20}x_9,\\
x_{26}x_4 &= -x_{21}x_9,\\
x_{26}x_8 &= x_{25}x_9,\\
x_{25}x_{21} &= x_{26}x_{20},\\
x_{20}^3 &= x_{48}x_4^3 + x_{36}x_8^3 - x_{20}^2x_8^2x_4.
\end{aligned}$$

Furthermore, the Steenrod algebra action on $H^*(BF_4;\mathbb{F}_3)$ is completely determined by the following relations:

	β	$\mathcal{P}^1$	$\mathcal{P}^3$	$\mathcal{P}^9$
x_4		$-x_8+x_4^2$		
x_8	x_9	x_8x_4	$x_{20}-x_8^2x_4$	
x_9			x_{21}	
x_{20}	x_{21}		$x_{20}(-x_8+x_4^2)$	$(x_{48}+x_{20}^2x_8)(-x_8+x_4^2)$ $+x_{36}(x_{20}+x_8^2x_4)+x_{26}x_{21}x_9$
x_{21}		x_{25}		$-x_{48}x_9+x_{36}x_{21}$
x_{25}	x_{26}			$x_{36}x_{25}-x_{26}^2x_9$
x_{26}				$x_{36}x_{26}$
x_{36}		$-x_{20}^2$	$x_{48}-x_{36}(x_8+x_4^2)x_4$ $+x_{20}^2(x_8+x_4^2)$	$-x_{48}x_{20}x_4+x_{48}(x_8^2+x_4^4)x_4^2-x_{36}^2$ $+x_{36}x_{20}(x_8+x_4^2)x_4^2-x_{36}(x_8^2+x_4^4)^2x_4$ $+x_{20}^2x_8(x_8^3+(x_8+x_4^2)^2x_4^2)$
x_{48}		x_{26}^2	$-x_{48}(x_8+x_4^2)x_4$	$-x_{48}x_{36}+x_{48}x_{20}(-x_8^2-x_8x_4^2+x_4^4)-x_{48}(x_8^2+x_4^4)^2x_4$

The important observation of Rector concerning the cohomology of BF_4 at the prime 3, is that the quotient of $H^*(BF_4;\mathbb{F}_3)$ by its radical $\sqrt{0}$, the ideal of all nilpotent elements, can be better understood that $H^*(BF_4;\mathbb{F}_3)$ itself and carries most of its information.

Recall that the radical of an algebra K, is defined as

$$\sqrt{0}=\{\,x\in K\mid x^r=0 \text{ for some integer } r\,\}.$$

It follows from Theorem 3.1 that the radical of $H^*(BF_4;\mathbb{F}_3)$ is the ideal generated by x_9, x_{21}, x_{25} and then $H^*(BF_4;\mathbb{F}_3)/\sqrt{0}$ is generated by classes

$$x_4, x_8, x_{20}, x_{26}, x_{36}, x_{48}$$

with the relations

$$\begin{aligned} x_4x_{26} = x_8x_{26} = x_{20}x_{26} = 0, \\ x_{20}^3 = x_{48}x_4^3 + x_{36}x_8^3 - x_{20}^2x_8^2x_4. \end{aligned}$$

Furthermore,

(1) The restriction map

$$\mathrm{res}_T\colon H^*(BF_4;\mathbb{F}_3) \longrightarrow H^*(BT;\mathbb{F}_3)^{W(F_4)}$$

factors through

$$\overline{\mathrm{res}}_T\colon H^*(BF_4;\mathbb{F}_3)/\sqrt{0} \longrightarrow H^*(BT;\mathbb{F}_3)^{W(F_4)},$$

mapping the classes x_4, x_8, x_{20}, x_{36}, and x_{48} to p_1, $\bar{p}_2$, $\bar{p}_5$, $\bar{p}_9$, and $\bar{p}_{12}$, respectively.

(2) The ideal generated by x_4, x_8, x_{20} is closed under the action of the Steenrod reduced power operations, hence, dividing out by this ideal we are left with a polynomial algebra $\mathbb{F}_3[x_{26}, x_{36}, x_{48}]$ with the following Steenrod algebra action:

$$\begin{array}{lll} \mathcal{P}^1x_{26} = 0 & \mathcal{P}^1x_{36} = 0 & \mathcal{P}^1x_{48} = x_{26}^2 \\ \mathcal{P}^3x_{26} = 0 & \mathcal{P}^3x_{36} = x_{48} & \mathcal{P}^3x_{48} = 0 \\ \mathcal{P}^9x_{26} = x_{36}x_{26} & \mathcal{P}^9x_{36} = -x_{36}^2 & \mathcal{P}^9x_{48} = -x_{48}x_{36} \end{array}$$

It turns out that this polynomial algebra is isomorphic, as algebras over the Steenrod algebra, to $P[u_1, u_2, u_3]^{SL_3(\mathbb{F}_3)}$ (see Example 2.4). Call φ the projection

$$\varphi\colon H^*(BF_4;\mathbb{F}_3)/\sqrt{0} \longrightarrow P[u_1, u_2, u_3]^{SL_3(\mathbb{F}_3)}.$$

It is a homomorphism of algebras over the Steenrod algebra of reduced powers with $\varphi(x_{26}) = L_3$, $\varphi(x_{36}) = Q_{3,2}$, $\varphi(x_{48}) = Q_{3,1}$, using the notation of Example 2.4.

(3) Similarly, if we further divide out by x_{26}, the quotient $P[x_{36}, x_{48}]$ can be identified, as algebras over the Steenrod algebra with the subalgebra of

$$P[x_1, x_2]^{GL_2(\mathbb{F}_3)} = P[Q_{2,1}, Q_{2,2}]$$

generated by $Q^3_{2,1}$ and $Q^3_{2,2}$ (see Example 2.3). We can then check that

$$\text{(3–1)}\quad H^*(BF_4;\mathbb{F}_3)/\sqrt{0} \cong P[x_{26},x_{36},x_{48}] \prod_{P[x_{36},x_{48}]} \frac{P[x_4,x_8,x_{20},x_{36},x_{48}]}{(x_{20}^3 = x_{48}x_4^3 + x_{36}x_8^3 - x_{20}^2x_8^2x_4)}$$

or, in other words, it fits in the pull-back diagram of algebras over the Steenrod algebra of reduced powers:

$$\text{(3–2)}\quad \begin{tikzcd} H^*(BF_4;\mathbb{F}_3)/\sqrt{0} \arrow[r, "\varphi"] \arrow[d, "\overline{\mathrm{res}}_T"'] & P[u_1,u_2,u_3]^{SL_3(\mathbb{F}_3)} \arrow[d, "\varsigma"] \\ H^*(BT;\mathbb{F}_3)^{W(F_4)} \arrow[r, "\varrho"] & P[x_1,x_2]^{GL_2(\mathbb{F}_3)}, \end{tikzcd}$$

where

$$\varrho\circ\overline{\mathrm{res}}_T(x_{36}) = \varrho(\bar{p}_9) = Q^3_{2,1} \qquad \varrho\circ\overline{\mathrm{res}}_T(x_{48}) = \varrho(\bar{p}_{12}) = Q^3_{2,2}$$
$$\varsigma(Q_{3,2}) = Q^3_{2,1} \qquad \varsigma(Q_{3,1}) = Q^3_{2,2},$$

while other generators are mapped trivially.

Our aim now is to extend the above diagram to one that captures the whole structure of $H^*(BF_4;\mathbb{F}_3)$. The main theoretical tool is the nil-localization functor for algebras over the Steenrod algebra (see Broto and Zarati [8] and Schwartz [16]).

Let $\mathcal{U}$ be the category of unstable modules over the Steenrod algebra and let $\mathcal{K}$ be the category of unstable algebras over the Steenrod algebra. A morphism $f\colon R\to S$ of $\mathcal{U}$ or $\mathcal{K}$ is called a nil-equivalence if the induced map $\mathrm{Hom}_{\mathcal{U}}(S,H^*V)\to\mathrm{Hom}_{\mathcal{U}}(R,H^*V)$ is a bijection for any elementary abelian p–group V, and $H^*V = H^*(BV,\mathbb{F}_p)$. Given an object K of $\mathcal{U}$, its nil-localization is another object $\mathcal{N}_{\mathcal{U}}^{-1}(K)$ of $\mathcal{U}$ together with a nil-equivalence $\mu_K\colon K\to\mathcal{N}_{\mathcal{U}}^{-1}(K)$ which is final among nil-equivalences with source K. If K is an object of $\mathcal{K}$, then so is $\mathcal{N}_{\mathcal{K}}^{-1}(K) = \mathcal{N}_{\mathcal{U}}^{-1}(K)$ and the universal map μ_K is a morphism of $\mathcal{K}$. We will say that K is reduced if μ_K is injective and nil-closed if μ_K is an isomorphism.

The Quillen map for a compact Lie group G, expressed as restriction from $H^*(BG,\mathbb{F}_p)$ to the inverse limit of cohomologies of elementary abelian p–subgroups E of G and morphisms induced by conjugation in G,

$$q_G\colon H^*(BG;\mathbb{F}_p) \longrightarrow \lim_{E\in\mathcal{E}_p(G)} H^*(BE;\mathbb{F}_p)$$

turns out to be the nil-localization of $H^*(BG, \mathbb{F}_p)$. Thus, the Adams conjecture can be rephrased by saying that for a compact and connected Lie group G and an odd prime p, $H^*(BG; \mathbb{F}_p)$ is a reduced object of $\mathcal{K}$.

By applying the nil-localization functor to Rector's diagram (3–2) we obtain our main result, that proves the conjecture of Adams for $G = F_4$ and $p = 3$.

Theorem 3.2 *There is a pull-back diagram:*

(3–3)
$$\begin{tikzcd} \mathcal{N}_{\mathcal{K}}^{-1}\big(H^*(BF_4; \mathbb{F}_3)\big) \arrow[r] \arrow[d] & H^*(BV_3; \mathbb{F}_3)^{SL_3(\mathbb{F}_3)} \arrow[d] \\ H^*(BV_4; \mathbb{F}_3)^{W(F_4)} \arrow[r] & H^*(BV_2; \mathbb{F}_3)^{GL_2(\mathbb{F}_3)} \end{tikzcd}$$

and the nil-localization $\mu\colon H^*(BF_4; \mathbb{F}_3) \to \mathcal{N}_{\mathcal{K}}^{-1}\big(H^*(BF_4; \mathbb{F}_3)\big)$ *is injective.*

The extension of diagram (3–2) to (3–3) requires the fact that the nil-localization of an object K of $\mathcal{K}$ coincides with that of $K/\sqrt{0}$. Notice, though, that the natural projection $K \to K/\sqrt{0}$ is not in general a morphism of $\mathcal{K}$. It might not commute with the action of the Bockstein operator. Indeed, in our case, x_{25} is in the radical of $H^*(BF_4; \mathbb{F}_3)$ but $\beta x_{25} = x_{26}$ is not nilpotent. In order to overcome this difficulty we introduce $\mathcal{K}'$, the full subcategory of objects of $\mathcal{K}$ concentrated in even degrees and the right adjoint functor $\widetilde{O}\colon \mathcal{K} \to \mathcal{K}'$ of the inclusion functor, described for any K of $\mathcal{K}$ as the subalgebra of even degree elements anihilated by the right ideal of the Steenrod algebra generated by the Bockstein operator (see Broto and Zarati [8] and Schwartz [16]). This adjoint pair provides a natural map $j\colon \widetilde{O}K \to K$, and the composition

$$\kappa_K\colon \widetilde{O}K \to K \to K/\sqrt{0}$$

is clearly a morphism of $\mathcal{K}'$. Moreover, it is a nil-equivalence. In fact, $j\colon \widetilde{O}K \to K$ is always injective and a nil-equivalence, so the kernel of κ_K is the radical of $\widetilde{O}K$. An element in the cokernel of κ_K is represented by an element of K, but the pth power of any element of K belongs to $\widetilde{O}K$, hence this cokernel is also nilpotent, so what we obtain is a diagram of nil-equivalences

$$\begin{tikzcd} \widetilde{O}K \arrow[r, "j"] \arrow[d, "\kappa_K"'] & K \arrow[d, "\mu_K"] \\ K/\sqrt{0} \arrow[r, "\mu_{K/\sqrt{0}}"'] & \mathcal{N}_{\mathcal{K}}^{-1}K \end{tikzcd}$$

Notice that a nil-equivalence between objects of $\mathcal{K}'$ is precisely an (F)–isomorphism in the sense of Quillen [14]. If K is a nil-closed object, then κ_K is an isomorphim and $j\colon K/\sqrt{0} \cong \widetilde{O}K \to K$ is the nil-localization.

Proof of Theorem 3.2 For an elementary abelian p–group V and G a subgroup of $GL(V)$, $H^*(BV,\mathbb{F}_p)$ is nil-closed and then

$$S(V^*) \cong H^*(BV,\mathbb{F}_p)/\sqrt{0} \cong \widetilde{O}H^*(BV,\mathbb{F}_p).$$

Since $\widetilde{O}$ commutes with inverse limits and the inverse limit of nil-closed objects is nil-closed, we also have that $S(V^*)^G \cong \widetilde{O}H^*(BV,\mathbb{F}_p)^G$ and the inclusion $S(V^*)^G \to H^*(BV,\mathbb{F}_p)^G$ is the nil-localization of $S(V^*)^G$.

Similarly, the inverse limit of a functor $c \in \mathcal{C} \mapsto H^*(V_c;\mathbb{F}_p)^{G_c} \in \mathcal{K}$ is nil-closed and $\widetilde{O}\lim_{c\in\mathcal{C}} H^*(V_c;\mathbb{F}_p)^{G_c} = \lim_{c\in\mathcal{C}} S(V_c^*)^{G_c}$, hence if $L = \lim_{c\in\mathcal{C}} S(V_c^*)^{G_c}$, then $\mathcal{N}_{\mathcal{K}}^{-1}L = \lim_{c\in\mathcal{C}} H^*(V_c;\mathbb{F}_p)^{G_c}$. This applies to the pull-back diagram (3–2) and proves that (3–3) in the statement of the theorem is again a pull-back diagram.

We will identify the composition of $\mu\colon H^*(BF_4;\mathbb{F}_3) \to \mathcal{N}_{\mathcal{K}}^{-1}\big(H^*(BF_4;\mathbb{F}_3)\big)$ with each of the maps in diagram (3–3) to the cohomology of an elementary abelian 3–subgroup.

(1) $H^*(BF_4;\mathbb{F}_3) \to H^*(BV_4;\mathbb{F}_3)^{W(F_4)}$. This map clearly factors as

$$H^*(BF_4;\mathbb{F}_3) \overset{\mathrm{res}_T}{\to} H^*(BT;\mathbb{F}_3)^{W(F_4)} \to H^*(BV_4;\mathbb{F}_3)^{W(F_4)}$$

and the kernel is the ideal of $H^*(BF_4;\mathbb{F}_3)$ generated by x_9, x_{21}, x_{25}, x_{26}.

(2) $\widehat{\varphi}\colon H^*(BF_4;\mathbb{F}_3) \to H^*(BV_3;\mathbb{F}_3)^{SL_3(\mathbb{F}_3)}$ is defined as the composition

$$H^*(BF_4;\mathbb{F}_3) \to \mathcal{N}_{\mathcal{K}}^{-1}(H^*(BF_4;\mathbb{F}_3)) \to H^*(BV_3;\mathbb{F}_3)^{SL_3(\mathbb{F}_3)}$$

obtained by applying the nil-localization functor to φ:

$$\begin{tikzcd}
\widetilde{O}H^*(BF_4;\mathbb{F}_3) \arrow[r] \arrow[d] & H^*(BV_3;\mathbb{F}_3)/\sqrt{0} \arrow[r, "\varphi"] \arrow[d] & S(V_3^*)^{SL_3(\mathbb{F}_3)} \arrow[d] \\
H^*(BF_4;\mathbb{F}_3) \arrow[r] & \mathcal{N}_{\mathcal{K}}^{-1}H^*(BF_4;\mathbb{F}_3) \arrow[r] & H^*(BV_3;\mathbb{F}_3)^{SL_3(\mathbb{F}_3)}
\end{tikzcd}$$

Notice that pth powers of even diamensional elements in an object K of $\mathcal{K}$ belong to $\widetilde{O}(K)$. In particular, $x_{26}^3, x_{36}^3, x_{48}^3$ belong to $\widetilde{O}H^*(BF_4;\mathbb{F}_3)$, thus they are mapped to $L_3^3, Q_{3,2}^3, Q_{3,1}^3 \in H^*(BV_3;\mathbb{F}_3)^{SL_3(\mathbb{F}_3)}$, respectively by $\widehat{\varphi}$.

Now, the other generators of $H^*(BF_4;\mathbb{F}_3)$, $x_{25},x_{21},x_{20},x_9,x_8,x_4$ are linked by Steenrod operations to x_{26} (see Theorem 3.1), hence they can not be in the kernel of $\widehat{\varphi}$.

The invariant ring $H^*(BV_3;\mathbb{F}_p)^{SL_3(\mathbb{F}_p)}$ is described in Example 2.4. If $p=3$, besides the polynomial generators L_3, $Q_{3,2}$ and $Q_{3,1}$, we have $m_3 = M_{1,2,3}$, $m_4 = \beta m_3 = M_{2,3}$, $m_8 = -\mathcal{P}^1 m_4 = M_{1,3}$, $m_{20} = \mathcal{P}^p m_8 = M_{1,2}$, $m_9 = \beta m_8 = M_3$, $m_{21} = \mathcal{P}^p m_9 = M_2$, and $m_{25} = \mathcal{P}^1 m_{21} = M_1$. Recall also that $\beta m_{25} = L_3$. Here the subindices of the lowercase m's indicate the degree in which they appear. It follows that $\widehat{\varphi}(x_4)$ can only be $\pm m_4$ and since $\widehat{\varphi}(x_{26}^3) = L_3^3$, it has to be $+m_4$, and

$$\widehat{\varphi}\colon H^*(BF_4;\mathbb{F}_23) \longrightarrow H^*(BV_3;\mathbb{F}_3)^{SL_3(\mathbb{F}_3)}$$

maps

$$\begin{array}{ccc} x_4 \mapsto m_4 & x_8 \mapsto m_8 & x_9 \mapsto m_9 \\ x_{20} \mapsto m_{20} & x_{21} \mapsto m_{21} & x_{25} \mapsto m_{25} \\ x_{26} \mapsto L_3 & x_{36} \mapsto Q_{3,2} & x_{48} \mapsto Q_{3,1} \end{array}$$

It is now routine to check that

$$\ker \widehat{\varphi} = (x_4^2, x_8^2, x_{20}^2, x_{20}x_8, x_{20}x_4, x_8x_4)$$

and that this ideal is contained in the subalgebra of $H^*(BF_4;\mathbb{F}_3)$ generated by x_4, x_8, x_{20}, x_{36}, x_{48} which is detected in $H^*(BT;\mathbb{F}_3)^{W(F_4)}$, hence the composition

$$H^*(BF_4;\mathbb{F}_3) \longrightarrow \mathcal{N}_{\mathcal{K}}^{-1}\big(H^*(BF_4;\mathbb{F}_3)\big) \longrightarrow H^*(BT;\mathbb{F}_3)^{W(F_4)} \times H^*(BV_3;\mathbb{F}_3)^{SL_3(\mathbb{F}_3)}$$

given by res_T and $\widehat{\varphi}$ is injective. □

The map $\widehat{\varsigma}\colon H^*(BV_3;\mathbb{F}_3)^{SL_3(\mathbb{F}_3)} \to H^*(BV_2;\mathbb{F}_3)^{GL_2(\mathbb{F}_3)}$, obtained as extension of ς in diagram (3–2), maps m_3 trivially (by degree reasons) and therefore m_4, m_8, m_9, m_{29}, m_{21}, m_{25}, L_3 are mapped trivially, too, and the image of $\widehat{\varsigma}$ is $P[x_{36}, x_{48}]$, which coincides with the image of ς. We can therefore express the mod 3 cohomology of BF_4 as the pull-back diagram

$$\begin{tikzcd} H^*(BF_4;\mathbb{F}_3) \arrow[r, "\widehat{\varphi}"] \arrow[d, "\mathrm{res}_T"'] & \operatorname{Im}\widehat{\varphi} \subset H^*(BV_3;\mathbb{F}_3)^{SL_3(\mathbb{F}_3)} \arrow[d, "\widehat{\varsigma}|_{\operatorname{Im}\widehat{\varphi}}"] \\ H^*(BT;\mathbb{F}_3)^{W(F_4)} \arrow[r, "\varrho"] & P[x_{36}, x_{48}] \end{tikzcd}$$

where $\operatorname{Im}\widehat{\varphi}$ is the subalgebra of $H^*(BV_3;\mathbb{F}_3)^{SL_3(\mathbb{F}_3)}$ generated by m_4, m_8, m_9, m_{29}, m_{21}, m_{25}, L_3, $Q_{3,2}$, $Q_{3,1}$, thus leaving in the cokernel only $\operatorname{Coker}\widehat{\varphi} \cong m_3P[Q_{3,2},Q_{3,1}]$, or, in other words,

$$H^*(BF_4;\mathbb{F}_3) \cong \operatorname{Im}\widehat{\varphi} \prod_{P[x_{36},x_{48}]} H^*(BT;\mathbb{F}_3)^{W(F_4)},$$

which we think is the correct way to understand this cohomology ring as algebra over the Steenrod algebra.

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